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A reformulation and applications of interfacial fluids with a free surface

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A non-local formulation, depending on a free spectral parameter, is presented governing two ideal fluids separated by a free interface and bounded above either by a free surface or by a rigid lid. This formulation is shown to be related to the Dirichlet–Neumann operators associated with the two-fluid equations. As an application, long wave equations are obtained; these include generalizations of the Benney–Luke and intermediate long wave equations, as well as their higher order perturbations. Computational studies reveal that both equations possess lump-type solutions, which indicate the possible existence of fully localized solitary waves in interfacial fluids with sufficient surface tension.

1. Introduction

In this paper, we present a new integral formulation of two ideal fluids separated by a free interface and bounded above by a free surface or rigid lid.

In recent years, there has been significant effort towards developing alternative formulations of the classic two-fluid equations. In Benjamin & Bridges (1997), the Euler equations governing two fluids separated by a free interface are developed in a Hamiltonian framework. This work generalizes the Hamiltonian formulation of water waves pioneered in Zakharov (1968) and extended in Craig & Sulem (1993) (see also Craig & Groves 1994; Craig, Schanz & Sulem 1997; Craig & Nicholls 2000; and Craig *et al.* 2002). In Craig, Guyenne & Kalisch (2005*a*), a Hamiltonian formulation of two fluids with a free interface and surface is obtained in terms of the interface and surface variables using Dirichlet–Neumann operators.

Here we generalize the integral formulation of water waves developed in Ablowitz, Fokas & Musslimani (2006) to the two-fluid system with a free surface. In particular, for the free surface problem we derive the following equations for the interface η, the surface β, $q \equiv \varphi(x, \eta, t)$, $Q \equiv \Phi(x, \eta, t)$ and $P \equiv \Phi(x, \beta + H, t)$, where φ and Φ denote the bottom and top velocity potentials (see figure 1):

$$\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \cosh(|k|\eta + |k|h)\eta_t \,\mathrm{d}x = i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \frac{\sinh(|k|\eta + |k|h)}{|k|}(k\cdot\nabla)q \,\mathrm{d}x, \tag{1.1}$$

$$\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|\beta)\beta_t \,\mathrm{d}x - \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|(\eta - H))\eta_t \,\mathrm{d}x$$
$$= -i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \frac{\cosh(|k|(\eta - H))}{|k|} (k\cdot\nabla)\, Q \,\mathrm{d}x + i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \frac{\cosh(|k|\beta)}{|k|} (k\cdot\nabla)\, P \,\mathrm{d}x, \tag{1.2}$$

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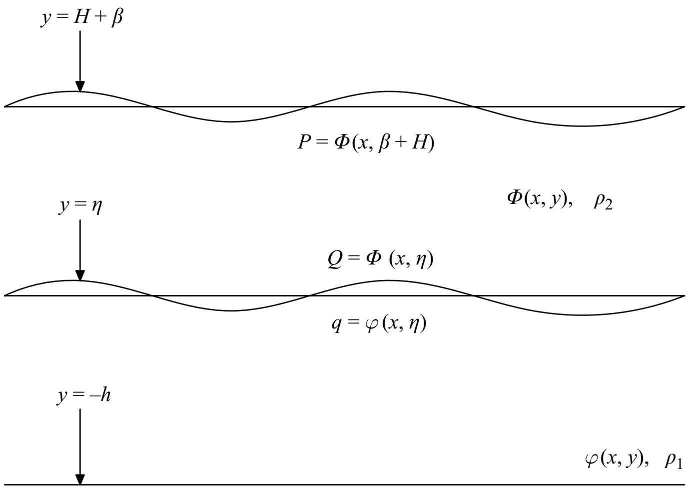

FIGURE 1. Two fluids with two free surfaces.

$$\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|(\beta + H))\beta_t \, \mathrm{d}x - \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|\eta)\eta_t \, \mathrm{d}x$$

$$= -i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \frac{\cosh(|k|\eta)}{|k|} (k \cdot \nabla) \, Q \, \mathrm{d}x + i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \frac{\cosh(|k|(\beta + H))}{|k|} (k \cdot \nabla) \, P \, \mathrm{d}x, \quad (1.3)$$

$$\rho_1 \left(q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2\left(1 + |\nabla \eta|^2\right)} \right)$$

$$- \rho_2 \left(Q_t + \frac{1}{2}|\nabla Q|^2 + g\eta - \frac{(\eta_t + \nabla Q \cdot \nabla \eta)^2}{2\left(1 + |\nabla \eta|^2\right)} \right) = \sigma_1 \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right), \quad (1.4)$$

$$P_t + \frac{1}{2}|\nabla P|^2 + g\beta - \frac{(\beta_t + \nabla P \cdot \nabla \beta)^2}{2\left(1 + |\nabla \beta|^2\right)} = \frac{\sigma_2}{\rho_2} \nabla \cdot \left(\frac{\nabla \beta}{\sqrt{1 + |\nabla \beta|^2}} \right). \quad (1.5)$$

In (1.1)–(1.5), $x = (x_1, x_2)$, $k = (k_1, k_2)$, $kx \equiv k \cdot x$ and the constants g, ρ_1, ρ_2, σ_1 and σ_2 denote gravity, density of the bottom fluid, density of the top fluid and surface tension at the interface η free surface $\beta + H$, respectively. Equations (1.4) and (1.5) are Bernoulli's equations, expressed in terms of the interface and surface variables η, β, q, Q and P. Equations (1.1)–(1.3) depend on a free spectral parameter k, and are referred to in this paper as the non-local spectral (NSP) equations. In Haut (2008), the following NSP formulation of the two-fluid problem with a fixed lid is derived from (1.1)–(1.5):

$$\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \cosh(|k|(\eta + h))\eta_t \, \mathrm{d}x = i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|(\eta + h)) \left(\frac{k}{|k|} \cdot \nabla \right) q \, \mathrm{d}x,$$

$$\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \cosh(|k|(\eta - H))\eta_t \, \mathrm{d}x = i \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx} \sinh(|k|(\eta - H)) \left(\frac{k}{|k|} \cdot \nabla \right) Q \, \mathrm{d}x,$$

$$\rho_1\left(q_t-\frac{1}{2}|\nabla q|^2+g\eta-\frac{(\eta_t+\nabla q\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)}\right)$$
$$-\rho_2\left(Q_t-\frac{1}{2}|\nabla Q|^2+g\eta-\frac{(\eta_t+\nabla Q\cdot\nabla\eta)^2}{2(1+|\nabla\eta|^2)}\right)=\sigma_1\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right).$$

We note that Fokas (cf. Fokas 2000 and Fokas 2007) has extensively used such integral formulations in his analysis of partial differential equations.

We show that the NSP equations (1.1)–(1.3) are equivalent to the kinematic conditions in the classic two-fluid Euler equations. In particular, we derive from (1.1)–(1.3) the series expansions of the Dirichlet–Neumann operators given in Craig *et al.* (2005*a*). We also use the adjoint properties of the Dirichlet–Neumann operators to obtain from (1.1)–(1.3) a dual system of non-local equations for the two-fluid kinematic conditions.

As an application of the NSP formulation, we derive in (2+1) dimensions new scalar long-wave reductions of two fluids with a free surface or rigid lid. Long-wave reductions of two-fluid systems with either a free surface or a rigid lid have been extensively studied (cf. the review Helfrich & Melville 2006). Notable model equations in the case of a rigid upper lid include the Korteweg–de Vries (KdV) equation (Benjamin 1966), the Benjamin–Ono (BO) equation (Benjamin 1967 and Ono 1975) and the intermediate long wave (ILW) equation (Joseph 1977; Kubota, Ko & Dobbs 1978; and Ablowitz & Clarkson 1991). Generalizations of the BO equation include a (2+1)-dimensional version derived in Ablowitz & Segur (1981) from continuously stratified fluids using a multiple-scale analysis, and a higher order BO equation derived in Matsuno (1992, 1994). In Choi & Camassa (1996), Boussinesq-type equations are obtained in (2+1) dimensions for both the free surface and the rigid lid cases, as well as a (2+1)-dimensional version of the ILW equation. The (1+1)-dimensional model equations are extended in Choi & Camassa (1999) under the sole assumption of long waves, i.e. no small assumption is made on the wave amplitude. In Craig *et al.* (2005*a*), series expansions for Dirichlet–Neumann operators are used to obtain in (1+1) dimensions higher order long-wave reductions in a variety of asymptotic regimes. A (1+1)-dimensional Boussinesq-type model is used in Bridges & Donaldson (2007) to study interfacial solitary waves in the rigid lid case. In Bona, Lannes & Saut (2008), Boussinesq-type equations are derived and rigorously analysed in (2+1) dimensions for the rigid lid case.

Here we obtain a new generalization of the (2+1)-dimensional Benney–Luke (BL) equation (Benney & Luke 1964) for interfacial fluids with a free surface or rigid lid, which we refer to in this paper as the ILW–BL equation; we also present a higher order ILW–BL equation. One aim in deriving the ILW–BL equation is the investigation of fully localized interfacial solitary waves. Such localized solutions are known to exist for the classical BL equation (see Berger & Milewski 2000 and Pego & Quintero 1999), and have motivated the existence proof of lump solutions to the full water wave equations with large enough surface tension (cf. Groves & Sun 2008). One advantage of the ILW–BL equation over Boussinesq-type models is that the equation is scalar, and the computation of solitary waves can be carried out in a straightforward way in (2+1) dimensions. In particular, we use a direct iteration method (cf. Ablowitz & Musslimani 2005) to compute fully localized solutions to the ILW–BL equation.

The two dispersion relations resulting from either a free surface or a rigid lid are qualitatively different in the long-wave regime (Craig *et al.* 2005*a*). Specifically, the dispersion relation resulting from a rigid lid gives one characteristic phase speed; in contrast, the dispersion relation resulting from a free surface gives, assuming long waves, characteristic phase speeds that are of different orders of magnitude unless both the upper and the lower fluids are shallow. This indicates that, while the ILW–BL equation fully captures the fluid dynamics resulting from a rigid lid, it only describes the fluid dynamics resulting from a free surface when either special initial conditions are given, or both the fluids are shallow.

A reduction of the ILW–BL equation gives a (2+1)-dimensional generalization of the ILW equation, which we refer to in this paper as the ILW–Kadomtsev–Petviashvili (ILW–KP) equation (see also Choi & Camassa 1999). We also obtain a new higher order ILW–KP equation, which reduces to a higher order (one-dimensional) ILW equation (Craig *et al.* 2005*a*) and a higher order BO equation (Matsuno 1992, 1994) upon neglecting y dependence and, in the latter case, taking an infinite bottom layer. The relationship between the ILW–KP equation and the ILW equation is analogous to that of the KP equation (Kadomtsev & Petviashvili 1970) and the KdV equation (Korteweg & deVries 1895). Unlike associated Boussinesq-type or BL-type equations, the ILW–KP equation has no small parameters in it, and, as such, is of basic importance. In particular, it is well known that the KP equation possesses rational solutions with a corresponding linear speed versus amplitude relationship (cf. Ablowitz & Clarkson 1991). We numerically compute analogous lump-type solutions to the ILW–KP equation, and show that the resulting speed versus amplitude relationship is nearly linear. The linearity of the speed–amplitude curve suggests that the ILW–KP equation could possess some of the remarkable properties exhibited by the KdV, KP, ILW and BO equations.

2. A non-local spectral reformulation of classic two-fluid equations

2.1. *A weak formulation of the classic two-fluid equations*

We recall the classic equations governing two ideal fluids separated by a free interface η and bounded above by a free surface β. It is assumed that the upper fluid is of density ρ_2 and the lower fluid is of density $\rho_1 > \rho_2$. The equations are given in terms of the interface and surface variables, and the velocity potentials φ and Φ associated with the upper and lower fluid domains, respectively:

$$\Delta\varphi = 0, \ \text{in } -h < y < \eta, \tag{2.1}$$

$$\varphi_y = 0, \ \text{on } y = -h, \tag{2.2}$$

$$\eta_t + \varphi_{x_1}\eta_{x_1} + \varphi_{x_2}\eta_{x_2} = \varphi_y, \ \text{on } y = \eta, \tag{2.3}$$

$$\Delta\Phi = 0, \ \text{in } \eta < y < H + \beta, \tag{2.4}$$

$$\eta_t + \Phi_{x_1}\eta_{x_1} + \Phi_{x_2}\eta_{x_2} = \Phi_y, \ \text{on } y = \eta, \tag{2.5}$$

$$\beta_t + \Phi_{x_1}\beta_{x_1} + \Phi_{x_2}\beta_{x_2} = \Phi_y, \ \text{on } y = H + \beta. \tag{2.6}$$

$$\rho_1\left(\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\eta\right) - \rho_2\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + g\eta\right) = \sigma_1\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right), \quad \text{on } y = \eta, \tag{2.7}$$

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + g\beta = \frac{\sigma_2}{\rho_2}\nabla\cdot\left(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right), \quad \text{on } y = H + \beta. \tag{2.8}$$

In the previous equations, the constants g, σ_1 and σ_2 denote gravity and the surface tensions associated with the interface and surface, respectively. We also assume that η, β, $|\nabla\varphi|$ and $|\nabla\Phi|$ vanish as $|x| \to \infty$.

We now obtain a weak formulation of (2.1)–(2.6). Specifically, the weak formulation consists of finding η, β, q, Q and P such for any suitable functions ψ and Ψ, defined in the lower fluid domain $D(\eta)$ and the upper fluid domain $D(\eta, \beta)$, and satisfying

$$\Delta\psi = 0, \quad \psi_y|_{y=-h} = 0, \quad \Delta\Psi = 0, \tag{2.9}$$

the following identities hold:

$$\int_{\mathbf{R}^2} (\psi|_{y=\eta})\eta_t \,\mathrm{d}x + \int_{\mathbf{R}^2} q(\psi_y - \nabla_x\psi \cdot \nabla_x\eta)|_{y=\eta} \,\mathrm{d}x = 0, \tag{2.10}$$

$$\begin{aligned} \int_{\mathbf{R}^2} (\Psi|_{y=\beta+H})\beta_t \,\mathrm{d}x - \int_{\mathbf{R}^2} \left(\Psi|_{y=\eta}\right) \eta_t \,\mathrm{d}x + \int_{\mathbf{R}^2} Q(\Psi_y - \nabla_x\Psi \cdot \nabla_x\eta)|_{y=\eta} \,\mathrm{d}x \\ - \int_{\mathbf{R}^2} P(\Psi_y - \nabla_x\Psi \cdot \nabla_x\beta)|_{y=\beta+H} \,\mathrm{d}x = 0. \end{aligned} \tag{2.11}$$

Equations (2.10) and (2.11), together with the Bernoulli equations (1.4) and (1.5) from the introduction, constitute a non-local system of equations describing the two-fluid system. In § (2.2), we will use a convenient basis to replace (2.10) and (2.11) with explicit integral equations. In § 2.3, we show that (2.10)–(2.11) and (2.1)–(2.6) are equivalent formulations of the kinematic conditions.

We now indicate how to derive (2.10) and (2.11) from the classic two-fluid equations (2.1)–(2.8). Specifically, assume that ψ and Ψ are suitably decaying functions defined in $D(\eta)$ and $D(\eta, \beta)$, respectively, that satisfy conditions (2.9).

To derive (2.10), apply Green's identity to φ and ψ in the domain $D(\eta)$ to get

$$0 = \int_{D(\eta)} (\varphi\,(\Delta\psi) - \psi\,(\Delta\varphi)) \,\mathrm{d}V = \int_{\partial D(\eta)} (\varphi\,(\nabla\psi \cdot \boldsymbol{n}) - \psi\,(\nabla\varphi \cdot \boldsymbol{n})) \,\mathrm{d}S = 0, \tag{2.12}$$

where $\boldsymbol{n}$ is the unit normal, $\mathrm{d}V$ is the volume measure, $\mathrm{d}S$ is the surface measure and $\partial D(\eta)$ is the boundary of the bottom fluid. Using

$$q = \varphi(x, \eta, t), \quad \eta_t = \varphi_y - \nabla_x\varphi \cdot \nabla_x\eta, \ \text{ on } y = \eta,$$

and the decaying boundary conditions in (2.12) yields (2.10), upon simplification.

Similarly, to derive (2.11), apply Green's identity to Φ and Ψ in $D(\eta, \beta)$ to get

$$\int_{\partial D(\eta,\beta)} \{\Phi\,(\nabla\Psi \cdot \boldsymbol{n}) - \Psi\,(\nabla\Phi \cdot \boldsymbol{n})\} \,\mathrm{d}S = 0, \tag{2.13}$$

where $\partial D(\eta, \beta)$ denotes the surface of the upper fluid domain $D(\eta, \beta)$. Using

$$\begin{aligned} &Q = \Phi(x, \eta, t), \quad P = \Phi(x, \beta + H, t), \\ &\eta_t = \Phi_y - \nabla_x\Phi \cdot \nabla_x\eta, \quad \text{on } y = \eta, \\ &\beta_t = \Phi_y - \nabla_x\Phi \cdot \nabla_x\beta, \quad \text{on } y = \beta + H, \end{aligned}$$

in (2.13) and simplifying gives (2.11).

2.2. *Derivation of the non-local spectral equations for two fluids*

We derive from (2.10) and (2.11) the NSP equations (1.3)–(1.5) given in the introduction.

First, we derive (1.3) from (2.10). To motivate the derivation, recall that the function ψ occurring in (2.10) satisfies

$$\Delta\psi(x, y) = 0 \qquad \psi_y(x, -h) = 0. \tag{2.14}$$

Letting $\hat{\psi}$ denote the Fourier transform of ψ in the x variable, conditions (2.14) can be expressed in terms of $\hat{\psi}$ as

$$\hat{\psi}_{yy}(k, y) = |k|^2\hat{\psi}(k, y), \quad \hat{\psi}_y(k, -h) = 0.$$

The solution to this differential equation is

$$\hat{\psi}(k, y) = \hat{\xi}(k)\cosh(|k|y + |k|h), \tag{2.15}$$

where $\hat{\xi}(k)$ is arbitrary. Reverting back to physical space,

$$\psi(x, y) = \int_{\mathbf{R}^2} \mathrm{e}^{ikx}\hat{\xi}(k)\cosh(|k|y + |k|h)\,\mathrm{d}k. \tag{2.16}$$

Therefore, any function ψ that satisfies (2.14) is formally a sum of basic functions ψ_k, where

$$\psi_k(x, y) = \mathrm{e}^{ikx}\cosh(|k|y + |k|h).$$

Since (2.10) is linear, it suffices to force (2.10) to hold for the parameterized family of functions ψ_k. Putting ψ_k into (2.10) gives

$$\int_{\mathbf{R}^2} \mathrm{e}^{ikx}\cosh(|k|\eta + |k|h)\eta_t\,\mathrm{d}x = \int_{\mathbf{R}^2} q\left(\mathrm{e}^{ikx}|k|\sinh(|k|\eta + |k|h) - i\mathrm{e}^{ikx}\cosh(|k|\eta + |k|h)\,(k\cdot\nabla)\,\eta\right)\mathrm{d}x. \tag{2.17}$$

Finally, using

$$\mathrm{e}^{ikx}|k|\sinh(|k|\eta + |k|h) - i\mathrm{e}^{ikx}\cosh(|k|\eta + |k|h)(k\cdot\nabla)\eta = -i\nabla\cdot\left(\mathrm{e}^{ikx}\frac{\sinh(|k|\eta + |k|h)}{|k|}k\right)$$

in (2.17) and integrating by parts gives us (1.1), upon noting that the boundary terms that come from integration by parts are zero when interpreted in the appropriate distributional sense. Indeed, for any smooth function $f(k)$ with $\mathrm{supp}(f) \subset (-L, L) \times (-L, L)$ (and suppressing t dependence),

$$\left\langle q\mathrm{e}^{ikx}\frac{k_j\sinh(|k|\eta + |k|h)}{|k|}, f\right\rangle = q(x)\int_{-L}^{L}\int_{-L}^{L}\mathrm{e}^{ikx}\frac{k_j\sinh(|k|\eta(x) + |k|h)}{|k|}f(k)\,\mathrm{d}k$$
$$= q(x)\int_{-L}^{L}\int_{-L}^{L}\mathrm{e}^{ikx}\frac{k_j\sinh(|k|h)}{|k|}f(k)\,\mathrm{d}k + \eta(x)q(x)\int_{-L}^{L}\int_{-L}^{L}\mathrm{e}^{ikx}\frac{k_j\cosh(\zeta_{x,k})}{|k|}f(k)\,\mathrm{d}k,$$

where $j = 1, 2$ and $|\zeta_{x,k}| \leqslant \sqrt{2}L|\eta(x)|$. The first integral in the second line of the previous equation goes to zero as $|x| \to \infty$, by the Riemann–Lebesgue lemma. The second integral also goes to zero as $|x| \to \infty$, since the integrand is bounded by a constant and $\eta \to 0$ as $|x| \to \infty$.

We use the same reasoning to derive equations (1.2)–(1.3) from (2.11). Specifically, it suffices that (2.11) holds for the basic functions

$$\Psi_k(x, y) = \mathrm{e}^{ikx}\sinh(|k|y - |k|H), \quad \Psi_k(x, y) = \sinh(|k|y)\mathrm{e}^{ikx}.$$

Taking $\Psi_k(x, y) = \mathrm{e}^{\mathrm{i}kx}\sinh(|k|y - |k|H)$ in (2.11), we get that

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx}\sinh(|k|\beta)\beta_t\,\mathrm{d}x - \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}kx}\sinh(|k|\eta - |k|H)\eta_t\,\mathrm{d}x \\
&\quad = -\int_{\mathbf{R}^2} Q\mathrm{e}^{\mathrm{i}kx}(|k|\cosh(|k|\eta - |k|H) - i\sinh(|k|\eta - |k|H)\,(k\cdot\nabla)\,\eta)\,\mathrm{d}x \\
&\qquad + \int_{\mathbf{R}^2} P\mathrm{e}^{\mathrm{i}kx}(|k|\cosh(|k|\beta) - i\sinh(|k|\beta)\,(k\cdot\nabla)\,\beta)\,\mathrm{d}x.
\end{aligned} \tag{2.18}
$$

Simplifying the previous equation gives us (1.2). Similarly, taking $\Psi_k(x, y) = \mathrm{e}^{\mathrm{i}kx}\sinh(|k|y)$ in (2.11) gives us (1.3).

It will be useful in our subsequent analysis to write (1.1)–(1.3) in operator notation. To do so, replace k by $-k$ in (1.1)–(1.3) and formally take the inverse Fourier transform to get

$$
\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\cosh(|k|\eta + |k|h)\eta_t\,\mathrm{d}x'\,\mathrm{d}k \\
&\quad = -i\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\sinh(|k|\eta + |k|h)\left(\frac{k_1}{|k|}q_{x_1'} + \frac{k_2}{|k|}q_{x_2'}\right)\mathrm{d}x'\,\mathrm{d}k,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\sinh(|k|\beta)\beta_t\,\mathrm{d}x'\,\mathrm{d}k - \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\sinh(|k|(\eta - H))\eta_t\,\mathrm{d}x'\,\mathrm{d}k \\
&\quad = \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\cosh(|k|(\eta - H))\left(i\frac{k_1}{|k|}Q_{x_1'} + i\frac{k_2}{|k|}Q_{x_2'}\right)\mathrm{d}x'\,\mathrm{d}k \\
&\qquad - \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\cosh(|k|\beta)\left(i\frac{k_1}{|k|}P_{x_1'} + i\frac{k_2}{|k|}P_{x_2'}\right)\mathrm{d}x'\,\mathrm{d}k,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\sinh(|k|(\beta + H))\beta_t\,\mathrm{d}x'\,\mathrm{d}k - \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\sinh(|k|\eta)\eta_t\,\mathrm{d}x'\,\mathrm{d}k \\
&\quad = \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\cosh(|k|\eta)\left(i\frac{k_1}{|k|}Q_{x_1'} + i\frac{k_2}{|k|}Q_{x_2'}\right)\mathrm{d}x'\,\mathrm{d}k \\
&\qquad - \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k(x-x')}\cosh(|k|(\beta(x', t) + H))\left(i\frac{k_1}{|k|}P_{x_1'} + i\frac{k_2}{|k|}P_{x_2'}\right)\mathrm{d}x'\,\mathrm{d}k.
\end{aligned}
$$

In operator notation, the previous equations become

$$A(\eta)\eta_t = B(\eta)q, \tag{2.19}$$

$$\begin{pmatrix} A_{11}(\eta) & A_{12}(\beta) \\ A_{21}(\eta) & A_{22}(\beta) \end{pmatrix}\begin{pmatrix} \eta_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} B_{11}(\eta) & B_{12}(\beta) \\ B_{21}(\eta) & B_{22}(\beta) \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}. \tag{2.20}$$

2.3. *Equivalence of classic and weak formulations*

In this section, we show that the weak formulation and equations (2.1)–(2.6) are equivalent.

To do so, it will be convenient to follow Craig *et al.* (2005*a*) and re-express (2.1)–(2.6) in terms of Dirichlet–Neumann operators acting on the interface and surface variables. Specifically, (2.1)–(2.3) can be written in terms of the Dirichlet–Neumann operator $G(\eta)$ as

$$\eta_t = G(\eta)q \equiv (\varphi_y - \nabla_x\varphi\cdot\nabla\eta)|_{y=\eta}, \tag{2.21}$$

where φ satisfies the Dirichlet problem

$$\Delta\varphi = 0, \text{ for } x \in \mathbf{R}^2 \text{ and } -h < y < \eta, \quad \varphi(x,\eta) = q, \quad \varphi_y(x,-h) = 0.$$

Similarly, (2.4)–(2.6) can be written in terms of the Dirichlet–Neumann operators $G_{ij}(\eta,\beta)$, $i, j = 1, 2$, as

$$\begin{aligned}\eta_t &= -G_{11}(\beta,\eta)Q - G_{12}(\beta,\eta)P \\ &\equiv (\Phi_{1y} - \nabla_x\Phi_1\cdot\nabla_x\eta)|_{y=\eta} + (\Phi_{2y} - \nabla_x\Phi_2\cdot\nabla_x\eta)|_{y=\eta},\end{aligned} \tag{2.22}$$

$$\begin{aligned}\beta_t &= G_{21}(\beta,\eta)Q + G_{22}(\beta,\eta)P \\ &\equiv (\Phi_{1y} - \nabla_x\Phi_1\cdot\nabla_x\beta)|_{y=\beta+H} + (\Phi_{2y} - \nabla_x\Phi_2\cdot\nabla_x\beta)|_{y=\beta+H},\end{aligned} \tag{2.23}$$

where Φ_1 and Φ_2 satisfy the Dirichlet problems

$$\begin{aligned}\Delta\Phi_1 &= 0, \quad \Phi_1(x,\eta) = Q, \quad \Phi_1(x,\beta+H) = 0. \\ \Delta\Phi_2 &= 0, \quad \Phi_2(x,\eta) = 0, \quad \Phi_2(x,\beta+H) = P.\end{aligned}$$

To show the equivalence of the two formulations, suppose that η, β, q, Q and P satisfy (2.10) and (2.11). We show that

$$\eta_t = -G_{11}(\eta,\beta)Q - G_{12}(\eta,\beta)P. \tag{2.24}$$

The other cases are similar.

Fix time t, and let $\tilde{Q}$ be an arbitrary, decaying function. In (2.11), let Ψ solve the Dirichlet problem

$$\Delta\Psi = 0 \text{ in } D(\eta,\beta), \quad \Psi(x,\eta) = \tilde{Q}, \quad \Psi(x,\beta+H) = 0.$$

Then (2.11) reduces to

$$\int_{\mathbf{R}^2} \tilde{Q}\eta_t\,\mathrm{d}x = \int_{\mathbf{R}^2} Q(\Psi_y - \nabla_x\Psi\cdot\nabla_x\eta)|_{y=\eta}\,\mathrm{d}x + \int_{\mathbf{R}^2} P(\Psi_y - \nabla_x\Psi\cdot\nabla_x\beta)|_{y=\beta+H}\,\mathrm{d}x. \tag{2.25}$$

Since

$$\begin{aligned}G_{11}(\eta,\beta)\tilde{Q} &= (-\Psi_y + \nabla_x\Psi\cdot\nabla_x\eta)|_{y=\eta}, \\ G_{21}(\eta,\beta)\tilde{Q} &= (-\Psi_y + \nabla_x\Psi\cdot\nabla_x\beta)|_{y=\beta+H},\end{aligned}$$

equation (2.25) can be written as

$$\int_{\mathbf{R}^2} \tilde{Q}\eta_t\,\mathrm{d}x = -\int_{\mathbf{R}^2} QG_{11}(\eta,\beta)\tilde{Q}\,\mathrm{d}x - \int_{\mathbf{R}^2} PG_{21}(\eta,\beta)\tilde{Q}\,\mathrm{d}x.$$

Using the identities

$$G_{11}(\eta,\beta)^* = G_{11}(\eta,\beta), \quad G_{21}(\eta,\beta)^* = G_{12}(\eta,\beta),$$

where $*$ denotes the adjoint (these follow from Green's identity), the previous equation implies that

$$\int_{\mathbf{R}^2} \tilde{Q}\,(\eta_t + G_{11}(\eta,\beta)Q + G_{12}(\eta,\beta)P)\,\mathrm{d}x = 0.$$

Since $\tilde{Q}$ is arbitrary, (2.24) follows.

2.4. *Dual integral equations*

In this section we derive a system of integral equations that are dual to (1.1)–(1.3). These are similar to an equation used in (Craig *et al.* 2005*b*) to analyse water wave propagation over a variable bottom.

The dual to (1.1) is given by the coupled system

$$\eta_t = -i\nabla \cdot \int_{\mathbf{R}^2} k\mathrm{e}^{\mathrm{i}k\cdot x} \frac{\sinh(|k|(\eta + h))}{|k|} \hat{\xi},\ \mathrm{d}k, \tag{2.26}$$

$$\int \mathrm{e}^{\mathrm{i}k\cdot x} \cosh((\eta + h)|k|))\hat{\xi}\ \mathrm{d}k = q. \tag{2.27}$$

Similarly, the dual to (1.2) and (1.3) is given by the coupled system

$$\eta_t = i\nabla \cdot \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \frac{\cosh(|k|\eta - |k|H)}{|k|} \widehat{\xi_1}\ \mathrm{d}k - i\nabla \cdot \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \frac{\cosh(|k|\eta)}{|k|} \widehat{\xi_2}\ \mathrm{d}k, \tag{2.28}$$

$$\beta_t = i\nabla \cdot \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \frac{\cosh(|k|\beta)}{|k|} \widehat{\xi_1}\ \mathrm{d}k - \nabla \cdot \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \frac{\cosh(|k|(\beta + H))}{|k|} \widehat{\xi_2}\ \mathrm{d}k, \tag{2.29}$$

$$Q = -\int_{\mathbf{R}^2} e^{\mathrm{i}k\cdot x} \sinh(|k|(\eta - H))\widehat{\xi_1}\ \mathrm{d}k + \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \sinh(|k|\eta)\widehat{\xi_2}\ \mathrm{d}k, \tag{2.30}$$

$$P = -\int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \sinh(|k|\beta)\widehat{\xi_1}\ \mathrm{d}k + \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}k\cdot x} \sinh(|k|(\beta + H))\widehat{\xi_2}\ \mathrm{d}k. \tag{2.31}$$

Equations (2.26)–(2.30) yield an alternative integral formulation of (2.1)–(2.6).

For the derivation, we use that equations (2.19)–(2.20) implicitly define expressions for the various Dirichlet–Neumann operators appearing in the previous section. Specifically, using equations (2.21)–(2.23) in (2.19)–(2.20) gives the following implicit expressions for $G(\eta)$ and $G_{ij}(\eta, \beta)$:

$$A(\eta)G(\eta) = B(\eta), \tag{2.32}$$

$$\begin{pmatrix} A_{11}(\eta) & A_{12}(\beta) \\ A_{21}(\eta) & A_{22}(\beta) \end{pmatrix} \begin{pmatrix} -G_{11}(\eta, \beta) & -G_{12}(\eta, \beta) \\ G_{21}(\eta, \beta) & G_{22}(\eta, \beta) \end{pmatrix} = \begin{pmatrix} B_{11}(\eta) & B_{12}(\beta) \\ B_{21}(\eta) & B_{22}(\beta) \end{pmatrix}. \tag{2.33}$$

Using the relationships

$$G(\eta)^* = G(\eta), \quad G_{11}(\eta, \beta)^* = G_{11}(\eta, \beta),$$
$$G_{22}(\eta, \beta)^* = G_{22}(\eta, \beta), \quad G_{12}(\eta, \beta)^* = G_{21}(\eta, \beta),$$

we take the adjoint of (2.32) and (2.33) to get

$$G(\eta)A(\eta)^* = B(\eta)^*, \tag{2.34}$$

$$\begin{pmatrix} -G_{11}(\eta, \beta) & G_{12}(\eta, \beta) \\ -G_{21}(\eta, \beta) & G_{22}(\eta, \beta) \end{pmatrix} \begin{pmatrix} A_{11}(\eta)^* & A_{21}(\beta)^* \\ A_{12}(\eta)^* & A_{22}(\beta)^* \end{pmatrix} = \begin{pmatrix} B_{11}(\eta)^* & B_{21}(\beta)^* \\ B_{12}(\eta)^* & B_{22}(\beta)^* \end{pmatrix}.$$

After some routine matrix manipulations, the previous matrix equation can be rewritten as

$$\begin{pmatrix} -G_{11}(\eta, \beta) & -G_{12}(\eta, \beta) \\ G_{21}(\eta, \beta) & G_{22}(\eta, \beta) \end{pmatrix} \begin{pmatrix} A_{11}(\eta)^* & -A_{21}(\beta)^* \\ -A_{12}(\eta)^* & A_{22}(\beta)^* \end{pmatrix} = \begin{pmatrix} B_{11}(\eta)^* & -B_{21}(\beta)^* \\ -B_{12}(\eta)^* & B_{22}(\beta)^* \end{pmatrix}. \tag{2.35}$$

Equations (2.34) and (2.35) implicitly give expressions for the Dirichlet–Neumann operators, and lead to the following alternative integral formulation of the two-fluid system (dual to (2.19) and (2.20)):

$$\eta_t = B(\eta)^* A(\eta)^{*-1} q, \tag{2.36}$$

$$\begin{pmatrix} \eta_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} B_{11}(\eta)^* & -B_{21}(\beta)^* \\ -B_{12}(\eta)^* & B_{22}(\beta)^* \end{pmatrix} \begin{pmatrix} A_{11}(\eta)^* & -A_{21}(\beta)^* \\ -A_{12}(\eta)^* & A_{22}(\beta)^* \end{pmatrix}^{-1} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{2.37}$$

The adjoints occurring in the previous two equations are explicitly given by

$$A(\eta)^* = \cosh(\eta(x)|D| + |D|H), \tag{2.38}$$

$$B(\eta)^* = D_1 \sinh(\eta(x)|D| + |D|H)\frac{D_1}{|D|} + D_2 \sinh(\eta(x)|D| + |D|H)\frac{D_1}{|D|}, \tag{2.39}$$

$$A_{11}(\eta)^* = -\sinh(\eta(x)|D| - |D|H), \tag{2.40}$$

$$A_{12}(\beta)^* = \sinh(\beta(x)|D|), \tag{2.41}$$

$$A_{21}(\eta)^* = -\sinh(\eta(x)|D|), \tag{2.42}$$

$$A_{22}(\beta)^* = \sinh(\beta(x)|D| - |D|H), \tag{2.43}$$

$$B_{11}(\eta)^* = -D_1 \cosh(\eta(x)|D| - |D|H)\frac{D_1}{|D|} - D_2 \cosh(\eta(x)|D| - |D|H)\frac{D_2}{|D|}, \tag{2.44}$$

$$B_{12}(\beta)^* = D_1 \cosh(\beta(x)|D|)\frac{D_1}{|D|} + D_2 \cosh(\beta(x)|D|)\frac{D_2}{|D|}, \tag{2.45}$$

$$B_{21}(\eta)^* = -D_1 \cosh(\eta(x)|D|)\frac{D_1}{|D|} - D_2 \cosh(\eta(x)|D|)\frac{D_2}{|D|}, \tag{2.46}$$

$$B_{22}(\beta)^* = D_1 \cosh(\beta(x)|D| + H|D|)\frac{D_1}{|D|} + D_2 \cosh(\beta(x)|D| + H|D|)\frac{D_2}{|D|}, \tag{2.47}$$

where

$$D = (D_1, D_2) = (-i\partial_{x_1}, -i\partial_{x_1}).$$

In the expressions for A, B, A^*_{ij} and B^*_{ij}, we have used standard pseudo-differential notation. For example, the action of B^*_{11} on a given function f is

$$(B_{11}(\eta)^* f)(x) = -i\partial_{x_1} \int_{\mathbf{R}^2} e^{ikx} \cosh(\eta(x)|k| - |k|H)\frac{k_1}{|k|}\hat{f}(k)\,dk$$
$$- i\partial_{x_2} \int_{\mathbf{R}^2} e^{ikx} \cosh(\eta(x)|k| - |k|H)\frac{k_2}{|k|}\hat{f}(k)\,dk.$$

As an example of how we obtained (2.38)–(2.47), we compute $A(\eta)^*$. Given suitable functions f and g,

$$\langle A(\eta)f, g\rangle = \int dx\,\overline{g(x)}\left(\iint dk\,dx' e^{ik(x-x')} \cosh(|k|(\eta(x') + H))f(x')\right),$$
$$\int dx' f(x')\left(\iint dk\,dx\, e^{ik(x-x')} \cosh(|k|(\eta(x') + H))\overline{g(x)}\right),$$
$$\int dx' f(x')\overline{\left(\iint dk\,dx\, e^{ik(x'-x)} \cosh(|k|(\eta(x') + H))g(x)\right)},$$
$$\int dx' f(x')\overline{\left(\int dk\, e^{ikx'} \cosh(|k|(\eta(x') + H))\hat{g}(k)\right)},$$
$$\langle f, A(\eta)^* g\rangle.$$

Employing similar techniques as in Craig *et al.* (2005*a*), we re-derive (2.28)–(2.31) from first principles (the derivation of (2.26)–(2.27) is similar). To do so, first note that if $\eta = \beta = 0$, then

$$\Phi(x, y, t) = -\int_{\mathbf{R}^2} e^{ik\cdot x} \frac{\sinh(|k|(y-H))}{\sinh(|k|H)} \hat{Q}(k, t)\, dk + \int_{\mathbf{R}^2} e^{ik\cdot x} \frac{\sinh(|k|y)}{\sinh(|k|H)} \hat{P}(k, t)\, dk$$

satisfies (2.4)–(2.6). In general, we look for a solution Φ of the form

$$\Phi(x, y, t) = -\int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|y - |k|H)\widehat{\xi_1}(k, t)\, dk + \int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|y)\widehat{\xi_2}(k, t)\, dk. \quad (2.48)$$

Then Φ satisfies (2.4)–(2.6) if

$$Q = -\int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|(\eta - H))\widehat{\xi_1}\, dk + \int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|\eta)\widehat{\xi_2}\, dk,$$
$$P = -\int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|\beta)\widehat{\xi_1}\, dk + \int_{\mathbf{R}^2} e^{ik\cdot x} \sinh(|k|(\beta + H))\widehat{\xi_2}\, dk.$$

Using (2.40)–(2.43), this can be written in operator notation as

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} A_{11}(\eta)^* & -A_{21}(\eta)^* \\ -A_{12}(\beta)^* & A_{22}(\beta)^* \end{pmatrix}^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} \quad (2.49)$$

With (2.49), we use (2.48) to calculate that

$$\begin{aligned}
\eta_t &= \Phi_y(x, \eta + H) - \nabla_x\Phi(x, \eta)\cdot\nabla_x\eta \\
&= -\int_{\mathbf{R}^2} e^{ik\cdot x}|k| \cosh(|k|\eta - |k|H)\widehat{\xi_1}\, dk + \int_{\mathbf{R}^2} e^{ik\cdot x}|k| \cosh(|k|\eta)\widehat{\xi_2}\, dk \\
&\quad + i\int_{\mathbf{R}^2} e^{ik\cdot x} (k\cdot\nabla_x\eta) \sinh(|k|\eta - |k|H)\widehat{\xi_1}\, dk - i\int_{\mathbf{R}^2} e^{ik\cdot x} (k\cdot\nabla_x\eta) \sinh(|k|\eta)\widehat{\xi_2}(k)\, dk \\
&= i\nabla_x\cdot\int_{\mathbf{R}^2} k e^{ik\cdot x} \frac{\cosh(|k|\eta - |k|H)}{|k|}\widehat{\xi_1}\, dk - i\nabla_x\cdot\int_{\mathbf{R}^2} k e^{ik\cdot x} \frac{\cosh(|k|\eta)}{|k|}\widehat{\xi_2}\cdot k
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta_t &= \Phi_y(x, \beta + H) - \nabla_x\Phi(x, \beta + H)\cdot\nabla_x\beta \\
&= i\nabla_x\cdot\int_{\mathbf{R}^2} k e^{ik\cdot x} \frac{\cosh(|k|\beta(x))}{|k|}\widehat{\xi_1}\, dk - i\nabla_x\cdot\int_{\mathbf{R}^2} k\ e^{ik\cdot x} \frac{\cosh(|k|(\beta + H))}{|k|}\widehat{\xi_2}\, dk.
\end{aligned}$$

Using (2.40)–(2.43), we can write the above two equations as

$$\begin{pmatrix} \eta_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} B_{11}(\eta)^* & -B_{21}(\eta)^* \\ -B_{12}(\beta)^* & B_{22}(\beta)^* \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \quad (2.50)$$

Combining (2.49) and (2.50) gives us (2.37).

3. Conservation laws and integral identities for two fluids

In Ablowitz *et al.* (2006), conservation laws and integral identities for water waves are derived from the non-local spectral formulation (see Benjamin & Olver 1982 for a systematic analysis of conservation laws and symmetries of classic water waves). We now derive the analogous conservation laws and integral identities from the NSP formulation.

Specifically, let $k \to \epsilon k$ in (1.1)–(1.5) and expand in ϵ. Setting the ϵ^0 coefficient to zero and taking $k_2 = 0$,

$$\partial_t \int_{\mathbf{R}^2} \left(\eta + \frac{\beta^2}{2H} - \frac{\eta^2}{2H}\right) \mathrm{d}x + \int_{\mathbf{R}^2} x_1 \left(\frac{P_{x_1}}{H} - \frac{Q_{x_1}}{H}\right) \mathrm{d}x = 0, \tag{3.1}$$

$$\partial_t \int_{\mathbf{R}^2} \left(\beta + \frac{\beta^2}{2H} - \frac{\eta^2}{2H}\right) \mathrm{d}x + \int_{\mathbf{R}^2} x_1 \left(\frac{P_{x_1}}{H} - \frac{Q_{x_1}}{H}\right) \mathrm{d}x = 0. \tag{3.2}$$

Subtracting (3.2) from (3.1) yields the equation

$$\partial_t \int_{\mathbf{R}^2} (\eta - \beta)\, dx = 0, \tag{3.3}$$

which is conservation of mass in the upper fluid domain.

Similarly, setting to zero the ϵ^1 coefficient coming from (1.1)–(1.5), and taking $k_2 = 0$,

$$\begin{aligned} &\partial_t \int_{\mathbf{R}^2} x_1 \left(-\eta + \frac{\eta^2}{2H} - \frac{\beta^2}{2H}\right) \mathrm{d}x \\ &\quad + \int_{\mathbf{R}^2} \left(\left(-\frac{{x_1}^2}{2H} + \frac{\beta^2}{2H} - \frac{H}{6}\right) P_{x_1} + \left(\frac{{x_1}^2}{2H} - \frac{\eta^2}{2H} - \frac{H}{3} + \eta\right) Q_{x_1}\right) \mathrm{d}x = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\partial_t \int_{\mathbf{R}^2} x_1 \left(-\beta + \frac{\eta^2}{2H} - \frac{\beta^2}{2H}\right) \mathrm{d}x \\ &\quad + \int_{\mathbf{R}^2} \left(\left(\frac{{x_1}^2}{2H} - \frac{\eta^2}{2H} + \frac{H}{6}\right) Q_{x_1} + \left(-\frac{{x_1}^2}{2H} + \frac{\beta^2}{2H} + \frac{H}{3} + \beta\right) P_{x_1}\right) \mathrm{d}x = 0, \end{aligned} \tag{3.5}$$

Now subtract (3.5) from (3.4) to get

$$\partial_t \int_{\mathbf{R}^2} x_1 (\beta - \eta) = \int_{\mathbf{R}^2} \left(\left(\frac{H}{2} + \beta\right) P_{x_1} + \left(-\eta + \frac{H}{2}\right) Q_{x_1}\right) \mathrm{d}x. \tag{3.6}$$

Similarly, setting $k_2 = 0$ in the ϵ^1 coefficient gives us

$$\partial_t \int_{\mathbf{R}^2} x_2 (\beta - \eta) = \int_{\mathbf{R}^2} \left(\left(\frac{H}{2} + \beta\right) P_{x_2} + \left(-\eta + \frac{H}{2}\right) Q_{x_2}\right) \mathrm{d}x. \tag{3.7}$$

Equations (3.6)–(3.7) represent the evolution of the center of mass; the right-hand side is the momentum of the fluid. Analogues of (3.6)–(3.7) are derived in Benjamin & Bridges (1997) for the case of two infinite fluid layers separated by a free interface.

Finally, setting the ϵ^2 term to zero yields the following virial-type formulae,

$$\begin{aligned} &\partial_t \int_{\mathbf{R}^2} \left(\frac{{x_j}^2}{2}(\beta - \eta) - \frac{H}{4}(\beta^2 + \eta^2) - \frac{1}{6}(\beta^3 + \eta^3)\right) \\ &\qquad = \int_{\mathbf{R}^2} x_j \left(\left(\frac{H}{2} + \beta\right) P_{x_j} + \left(-\eta + \frac{H}{2}\right) Q_{x_j}\right) \mathrm{d}x, \end{aligned} \tag{3.8}$$

where $j = 0, 1$. There are no known analogues of (3.8) in the literature.

4. Long-wave reductions of two fluids and solitary waves

4.1. *Non-dimensionalization of NSP equations*

We now non-dimensionalize (1.1)–(1.5). To do so, define the non-dimentional variables x', t', η', β', q', Q' and P' by

$$x_1' = \frac{x_1}{l}, \quad x_2' = \gamma\frac{x_2}{l}, \quad t' = \frac{\sqrt{gH}}{l}t, \quad a\eta' = \eta, \quad a\beta' = \beta,$$
$$q' = \frac{agHq}{\sqrt{gH}}, \quad Q' = \frac{aglQ}{\sqrt{gH}}, \qquad P' = \frac{aglP}{\sqrt{gH}},$$

where l is a characteristic wavelength, a is the characteristic wave amplitude and γ is a non-dimensional parameter. Then (1.1)–(1.5) become, after dropping primes,

$$\int_{\mathbf{R}^2} \mathrm{e}^{-\mathrm{i}kx}\cosh(|k|(\mu\epsilon\eta+\alpha))\eta_t\,\mathrm{d}x = -i\int_{\mathbf{R}^2}\mathrm{e}^{\mathrm{i}kx}\frac{\sinh(|k|(\mu\epsilon\eta+\alpha))}{|k|}(k\cdot\nabla)q\,\mathrm{d}x, \tag{4.1}$$

$$\mu\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\sinh(|k|\mu\epsilon\beta)\beta_t\,\mathrm{d}x - \mu\int_{\mathbf{R}^2}\mathrm{e}^{\mathrm{i}kx}\sinh(|k|(\mu\epsilon\eta-\mu))\eta_t\,\mathrm{d}x$$
$$= i\int_{\mathbf{R}^2}\mathrm{e}^{\mathrm{i}kx}\frac{\cosh(|k|(\mu\epsilon\eta-\mu))}{|k|}(k\cdot\nabla)\,Q\,\mathrm{d}x - i\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\frac{\cosh(|k|\mu\epsilon\beta)}{|k|}(k\cdot\nabla)\,P\,\mathrm{d}x, \tag{4.2}$$

$$\mu\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\sinh(|k|(\mu\epsilon\beta+\mu))\beta_t\,\mathrm{d}x - \mu\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\sinh(|k|\mu\epsilon\eta)\eta_t\,\mathrm{d}x$$
$$= -i\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\frac{\cosh(|k|\mu\epsilon\eta)}{|k|}(k\cdot\nabla)\,Q\,\mathrm{d}x + i\int_{\mathbf{R}^2}\mathrm{e}^{-\mathrm{i}kx}\frac{\cosh(|k|(\mu\epsilon\beta+\mu))}{|k|}(k\cdot\nabla)\,P\,\mathrm{d}x, \tag{4.3}$$

$$\left(\mu q_t + \frac{1}{2}\mu\epsilon^2|\nabla q|^2 + \eta - \mu\epsilon^2\frac{(\eta_t+\mu\epsilon\nabla q\cdot\nabla\eta)^2}{2\left(1+\epsilon^2\mu^2|\nabla\eta|^2\right)}\right)$$
$$-\rho\left(Q_t + \frac{1}{2}\epsilon|\nabla Q|^2 + \eta - \mu\epsilon\frac{(\eta_t+\epsilon\nabla Q\cdot\nabla\eta)^2}{2\left(1+\epsilon^2\mu^2|\nabla\eta|^2\right)}\right) = \mu^2\sigma_1\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+\epsilon^2\mu^2|\nabla\eta|^2}}\right), \tag{4.4}$$

$$P_t + \frac{1}{2}\epsilon|\nabla P|^2 + \beta - \frac{(\beta_t+\epsilon\nabla P\cdot\nabla\beta)^2}{2\left(1+\epsilon^2\mu^2|\nabla\beta|^2\right)} = \mu^2\sigma_2\nabla\cdot\left(\frac{\nabla\beta}{\sqrt{1+\epsilon^2\mu^2|\nabla\beta|^2}}\right). \tag{4.5}$$

In the previous equations,

$$\nabla = \left(\partial_{x_1}, \gamma\partial_{x_2}\right), \quad k = (k_1, \gamma k_2), \quad x = (x_1, x_2/\gamma),$$
$$\epsilon = \frac{a}{H} \quad \mu = \frac{H}{l}, \quad \alpha = \frac{h}{l}, \quad \tilde{\sigma_1} = \frac{\sigma_1}{\rho g H^2}, \quad \tilde{\sigma_2} = \frac{\sigma_2}{\rho g H^2}, \quad \rho = \frac{\rho_2}{\rho_1}.$$

We assume that μ, and γ^2 are $O(1)$, and that α is $O(1)$ or larger. We initially make no assumption on the size of ϵ; this added generality will allow us to derive a (2+1)-dimensional analogue of the 'fully nonlinear' Boussinesq-type equations derived in Choi & Camassa (1999). We also assume that σ_1 and σ_2 are of order of $O(1/\mu)$, which will allow us compute lumps to the reduced equations.

4.2. *A Benney–Luke type equation for interfacial fluids with a free surface or rigid lid*

In this section, we will derive from (4.1)–(4.5) (interfacial waves with a free surface) the following BL-type equation for interfacial waves with a free surface:

$$Q_{tt} - c_1^2 Q_{x_1x_1} - \gamma^2 c_1^2 Q_{x_2x_2} + \epsilon(2Q_{x_1}Q_{x_1t} + Q_t Q_{x_1x_1}) + \mu(-i\rho c_1^2 \coth(D_1\alpha) Q_{x_1x_1x_1} + \rho\left(c_1^4\tilde{\sigma}_2 + \tilde{\sigma}_1\right) Q_{x_1x_1x_1x_1}) = 0, \quad (4.6)$$

where $c_1^2 = (1-\rho)$. We refer to (4.6) as the ILW–BL equation. In Haut (2008), the following version of (4.6) is derived for interfacial waves with a fixed lid:

$$Q_{tt} - c_0^2 Q_{x_1x_1} - \gamma^2 c_0^2 Q_{x_2x_2} + \epsilon(2Q_{x_1}Q_{x_1t} + Q_t Q_{x_1x_1}) + \mu\left(-i\frac{c_0^2}{\rho}\coth(D_1\alpha) Q_{x_1x_1x_1} + \frac{\tilde{\sigma}}{\rho} Q_{x_1x_1x_1x_1}\right) = 0, \quad (4.7)$$

where $c_0^2 = (1/\rho - 1)$ and $\tilde{\sigma}$ is the non-dimensional surface tension associated with the interface.

In (4.6), if we make the change of variables

$$\xi = x_1 - c_1 t, \quad \tau = \epsilon t, \quad y = x_2,$$

and set $\gamma^2 = \mu = \epsilon$, we also get, to leading order, the following (2+1)-dimensional version of the ILW equation (see also Choi & Camassa 1996 for the zero surface tension case):

$$2c_1 w_{\xi\tau} + c_1^2 w_{yy} + 3c_1(w_\xi w)_\xi + i c_1^2 \rho \coth(\alpha D_1) w_{\xi\xi\xi} - \rho\left(c_1^4\tilde{\sigma}_2 + \tilde{\sigma}_1\right) w_{\xi\xi\xi\xi}, \quad (4.8)$$

where $w = Q_\xi$. We refer to (4.8) as the ILW–KP equation. The following ILW–KP equation can be derived for the rigid lid case (see Choi & Camassa 1996; Haut 2008):

$$2c_0 w_{\xi\tau} + c_0^2 w_{yy} + 3c_0(w_\xi w)_\xi + i\frac{c_0^2}{\rho}\coth(\alpha D_1) w_{\xi\xi\xi} - \frac{\tilde{\sigma}}{\rho} w_{\xi\xi\xi\xi}. \quad (4.9)$$

We now derive (4.6). Expanding (4.1), (4.2), (4.4) and (4.5) in μ and γ^2 and taking the inverse Fourier transform,

$$q_{x_1} = i\coth(\alpha D_1)\eta_t + o(1), \quad (4.10)$$

$$P_{x_1x_1} - Q_{x_1x_1} + \gamma^2\left(P_{x_2x_2} - Q_{x_2x_2}\right) = o(\gamma^2, \mu), \quad (4.11)$$

$$(1-\rho)\eta - \rho Q_t = \mu\tilde{\sigma}_1\eta_{x_1x_1} - \mu q_t + \epsilon\frac{\rho}{2}\left(Q_{x_1}^2\right) + \epsilon\gamma^2\frac{\rho}{2}\left(Q_{x_2}^2\right) + o(\gamma^2, \mu), \quad (4.12)$$

$$P_t + \beta + \frac{1}{2}\epsilon P_{x_1}^2 - \mu\tilde{\sigma}_2\beta_{x_1x_1} = o(\gamma^2, \mu). \quad (4.13)$$

In (4.10), $D_1 = -i\partial_{x_1}$. Similarly, multiply (4.2) by $\cosh(|k|\mu)$, subtract it from (4.3), and divide by $\sinh(|k|\mu)$. Expanding the result in μ and γ^2,

$$\eta_t = \beta_t + Q_{x_1x_1} + \gamma^2 Q_{x_2x_2} + \epsilon(P_{x_1}\beta_{x_1} - \eta_{x_1}Q_{x_1} + \beta P_{x_1x_1} - \eta Q_{x_1x_1}) + o(\gamma^2, \mu). \quad (4.14)$$

Equation (4.6) now follows from (4.10)–(4.14). In detail, recursively solve (4.13) for β in terms of P:

$$\beta = -P_t - \frac{1}{2}\epsilon P_{x_1}^2 - \mu\tilde{\sigma}_2 P x_1 x_1 t + o(\gamma^2, \mu, \epsilon). \quad (4.15)$$

Similarly, from (4.11) we get

$$Q = P + o(\gamma^2, \mu, \epsilon). \quad (4.16)$$

Using (4.15) and (4.16) in (4.14),

$$\eta_t = Q_{x_1x_1} - Q_{tt} + \gamma^2 Q_{x_2x_2} - \mu\tilde{\sigma}_1 Q_{x_1x_1tt} \\ - \epsilon\left(\eta_{x_1}Q_{x_1} + \eta Q_{x_1x_1} + 2Q_{x_1}Q_{x_1t} + Q_t Q_{x_1x_1}\right) + o(\gamma^2, \mu, \epsilon). \quad (4.17)$$

Now differentiate (4.12) with respect to x_1 and use (4.10) to express q_{x_1} in terms of η_t:

$$(1-\rho)\eta_{x_1} = \rho Q_{x_1t} + \mu\tilde{\sigma}_1\eta_{x_1x_1x_1} - \mu i\ \coth(\alpha D_1)\eta_{tt} + \epsilon\frac{\rho}{2}\left(Q_{x_1}^2\right)_{x_1} + o(\epsilon, \mu, \gamma^2). \quad (4.18)$$

Finally, differentiate (4.17) and (4.18) with respect to x_1 and t, respectively. Equating the resulting equations for η_{x_1t} and repeatedly using

$$\eta = \rho/(1-\rho)Q_t + o(1), \quad \eta_t = Q_{x_1x_1} - Q_{tt} + o(1), \quad Q_{tt} = (1-\rho)Q_{x_1x_1} + o(1),$$

gives (4.6), upon simplification.

Carrying out the above procedure to the next order in ϵ, we also get the following higher order ILW–BL equation, where $T = \coth(\alpha D_1)$, $x = x_1$, and $y = x_2$:

$$\begin{aligned}
&Q_{tt} - c_1^2 Q_{xx} \\
&\quad + \epsilon\big(- i\rho c_1^2 T\left(Q_{xxx}\right) - c_1^2 Q_{yy} + 2Q_x Q_{xt} + \left(c_1^2 + \rho\right) Q_t Q_{xx} + \rho\left(\sigma_2 c_1^4 + \sigma_1\right) Q_{xxxx}\big) \\
&\quad + \epsilon^2\Big(5i\rho T\left(Q_{xt}Q_{xx}\right) + 3i\rho T\left(Q_x Q_{xxt}\right) + 2i\rho T\left(Q_t Q_{xxx}\right) - \frac{1}{2}(\rho-1)\rho T^2(Q_{xxyy}) \\
&\quad - \frac{3}{2} i\rho c_1^2 T(Q_{xyy}) + \rho(2\rho-1)c_1^2 T^2\left(Q_{xxxx}\right) + i\rho\left(3\rho\sigma_2 c_1^4 + (3\rho-1)\sigma_1\right) T\left(Q_{xxxxx}\right) \\
&\quad + 2Q_y Q_{yt} + \left(c_1^2 + \rho\right) Q_t Q_{yy} - i\rho\left(T Q_{xxt}\right) Q_x - i\rho\left(T Q_{xt} Q_{xx}\right) + \frac{3}{2}\left(c_1^2 + \rho\right) Q_x^2 Q_{xx} \\
&\quad - \frac{2\rho\left(\sigma_2 c_1^4 + 2\sigma_1\right) Q_{xx}}{c_1^2} Q_{xxt} + 2\rho\left(\sigma_2 c_1^4 + \sigma_1\right) Q_{xxyy} - \frac{\rho\left(3\sigma_2 c_1^4 + 4\sigma_1\right) Q_{xt}}{c_1^2} Q_{xxx} \\
&\quad - \frac{\rho\sigma_1 Q_x}{c_1^2} Q_{xxxt} + \frac{1}{6} c_1^2\left((6\rho-3)c_1^2 - 3\rho+1\right) Q_{xxxx} - \frac{\rho\left(c_1^2+\rho\right)\left(\sigma_2 c_1^4 + \sigma_1\right) Q_t}{c_1^2} Q_{xxxx} \\
&\quad + \frac{\rho^2\left(\rho\sigma_2^2 c_1^6 - 2\sigma_1\sigma_2 c_1^4 - \sigma_1^2\right)}{c_1^2} Q_{xxxxxx}\Big) = O(\epsilon^3).
\end{aligned}$$

Similarly, here is the higher order ILW–KP equation:

$$\begin{aligned}
&2c_1 Q_{\xi\tau} + c_1^2 Q_{yy} + 3c_1 Q_\xi Q_{\xi\xi} + i\rho c_1^2 T(Q_{\xi\xi\xi}) - \rho\left(\sigma_2 c_1^4 + \sigma_1\right) Q_{\xi\xi\xi\xi} \\
&\quad + \epsilon\Big(5i\rho c_1 T(Q_{\xi\xi}^2) + 5i\rho c_1 T(Q_\xi Q_{\xi\xi\xi}) + \frac{3}{2} i\rho c_1^2 T(Q_{\xi yy}) - \frac{1}{2}\rho c_1^2 T^2(Q_{\xi\xi yy}) \\
&\quad + (1-2\rho)\rho c_1^2 T^2(Q_{\xi\xi\xi\xi}) - i\rho\left(3\rho\sigma_2 c_1^4 + (3\rho-1)\sigma_1\right) T(Q_{\xi\xi\xi\xi\xi}) - Q_{\tau\tau} \\
&\quad - i\rho c_1(T Q_{\xi\xi\xi})Q_\xi + c_1 Q_{yy} Q_\xi - 2Q_\xi Q_{\xi\tau} + 2c_1 Q_y Q_{\xi y} - i\rho c_1(T Q_{\xi\xi})Q_{\xi\xi} \\
&\quad - Q_\tau Q_{\xi\xi} - \frac{3}{2} Q_\xi^2 Q_{\xi\xi} - 2\rho\left(\sigma_2 c_1^4 + \sigma_1\right) Q_{\xi\xi yy} - \frac{\rho\left(5\sigma_2 c_1^4 + 8\sigma_1\right) Q_{\xi\xi}}{c_1} Q_{\xi\xi\xi} \\
&\quad + \frac{1}{3}(3\rho^2 - 3\rho + 1)c_1^2 Q_{\xi\xi\xi\xi} - \frac{\rho\left(\sigma_2 c_1^4 + 2\sigma_1\right) Q_\xi}{c_1} Q_{\xi\xi\xi\xi} \\
&\quad + \frac{\rho^2\left(-\rho\sigma_2^2 c_1^6 + 2\sigma_1\sigma_2 c_1^4 + \sigma_1^2\right)}{c_1^2} Q_{\xi\xi\xi\xi\xi\xi}\Big) = O(\epsilon^2).
\end{aligned}$$

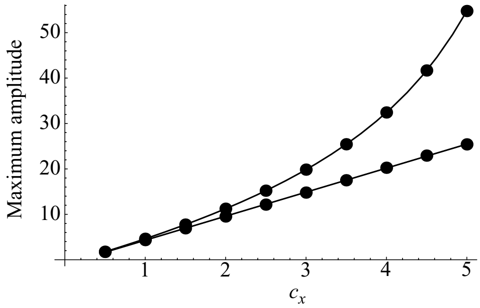

FIGURE 2. The speed versus maximum amplitude relationships for (4.19) and (4.20) when $\epsilon = 1/10$ and $c=(1/\sqrt{2})-c_x\epsilon$, with $c_x = .5, 1, 1.5, \ldots, 5$. Lower points correspond to (4.20), while upper points correspond to (4.19).

The term $Q_{\tau\tau}$ in the higher order ILW–KP equation can be asymptotically replaced by

$$Q_{\tau\tau} = -\frac{c_1}{2}\int_{-\infty}^{x} Q_{yy\tau}\,\mathrm{d}x - \frac{3}{4}c_1(Q_\xi^2)_\tau - i\rho\frac{c_1}{2}T(Q_{\xi\xi\tau}) + \frac{\rho}{2c_1}\left(\sigma_2 c_1^4 - \sigma_1\right) Q_{\xi\xi\xi\tau} + O(\epsilon).$$

4.3. *Lump solutions of the ILW–BL and ILW–KP equations*

We numerically investigate lump solutions of the ILW–BL equation (4.6), and compute the resulting speed versus maximum amplitude curve. We also compute lumps to the ILW–KP equation (4.9) assuming intermediate and deep bottom fluid layers ($\alpha = 1$ and $\alpha = 10$).

In (4.6), we move to a coordinate system travelling with velocity c in the x_1-direction (for convenience, we take zero velocity in the x_2-direction). By taking $\epsilon = \mu = \gamma^2$, $\rho = 1/2$, $\tilde{\sigma_1} = 1$, $\tilde{\sigma_2} = 0$, and $\alpha = 1$, we obtain

$$\left(\frac{1}{2} - c^2\right) w_{x_1x_1} + \frac{1}{2}\epsilon w_{x_2x_2} + \frac{3c}{2}\epsilon(w^2)_{x_1x_1} - \frac{1}{2}\epsilon w_{x_1x_1x_1x_1} + \frac{1}{4}\epsilon i\,\coth(D_1) w_{x_1x_1x_1} = 0, \tag{4.19}$$

where $w = Q_{x_1}$. Assuming that $c = \sqrt{1/2} - c_x\epsilon$, (4.19) becomes, to leading order, the ILW–KP equation

$$\sqrt{2}c_x w_{x_1x_1} + \frac{1}{2}w_{x_2x_2} + \frac{3}{2\sqrt{2}}(w^2)_{x_1x_1} - \frac{1}{2}w_{x_1x_1x_1x_1} + \frac{1}{4}i\,\coth(D_1) w_{x_1x_1x_1} = 0. \tag{4.20}$$

We use the spectral renormalization (SPRZ) method (Ablowitz & Musslimani 2005) to find lump solutions to (4.19) and (4.20) when $\epsilon = 1/10$ and $\epsilon = 1/100$. The SPRZ method is explained in Appendix B. In (4.19) and (4.20), we take $c = 1 - \epsilon c_x$ and $c_x = .5, 1, 1.5, \ldots, 5$.

Figure 2 shows the resulting speed versus maximum amplitude relationship when $\epsilon = 1/10$. The upper points in figure 2 correspond to solutions of the ILW–BL equation (4.19), while the lower points correspond to solutions of the ILW–KP equation (4.20). Each upper dot represents a point of the form (c_x, w_{max}), where w_{max} is the maximum amplitude of the solution $w(x, y)$ to (4.19) corresponding to $c = 1 - \epsilon c_x$. Similarly, each lower dot represents a point of the form (c_x, w_{max}), w_{max} is the maximum amplitude of the solution to (4.20) corresponding to c_x. We see from figure 2 that the speed–amplitude curve for the ILW–KP equation is a straight line. In contrast, the speed–amplitude curve for the ILW–BL equation begins to deviate markedly with

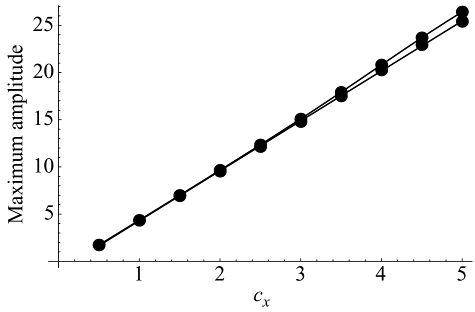

FIGURE 3. The speed versus maximum amplitude relationship for (4.19) and (4.20) when $\epsilon = 1/100$ and $c = (1/\sqrt{2}) - c_x\epsilon$, with $c_x = .5, 1, 1.5, \ldots, 5$. Lower points correspond to (4.20), while upper points correspond to (4.19)

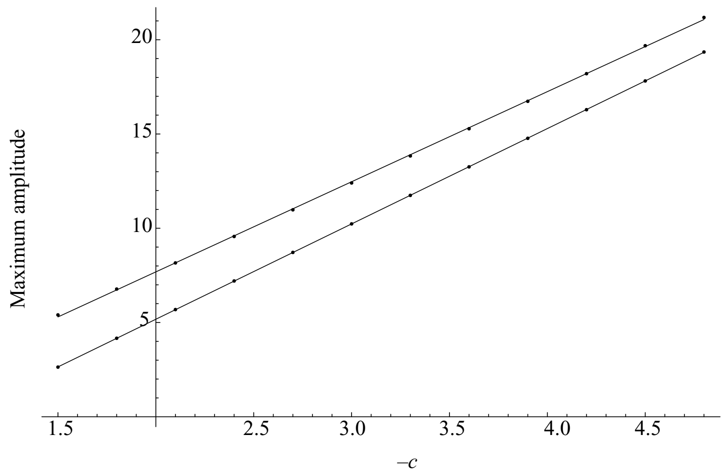

FIGURE 4. Maximum amplitude versus speed $-c$ for $\alpha = 1$ (lower line) and $\alpha = 10$ (upper line), i.e. the ILW–KP and ILW–BO regimes, respectively.

strong curvature from that of the ILW–KP (straight line) curve for speeds c away from $\sqrt{1/2}$.

Figure 3 shows the resulting speed versus maximum amplitude relationship when $\epsilon = 1/100$. As in figure 2, upper points in figure 3 correspond to solutions of (4.19), while the lower points correspond to solutions of (4.20). We see from figure 3 that the speed–amplitude curve for the ILW–BL equation approaches that of the ILW–KP equation for $\epsilon = 1/100$.

In a similar manner, we now compute lump solutions to the ILW–KP equation (4.9) for $\alpha = 1$ and $\alpha = 10$. In (4.9), we take $\rho = 1/2$, $\tilde{\sigma} = 1$, and pass to a travelling coordinate system moving with velocity c in the x_1-direction. Figure 4 displays the resulting speed versus amplitude relationship, which is linear in both cases. Note that the horizontal axis in figure 4 is $-c$. Figures 5 and 6 display the x cross-sections $w(x, 0)$ and y cross-sections $w(0, y)$, respectively, for a typical speed $c = -1.5$ and $\alpha = 1$.

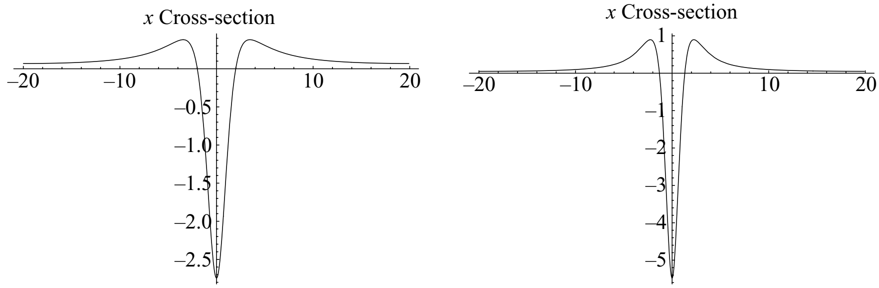

FIGURE 5. ILW–KP x cross-sections $w(x, 0)$ for $\alpha = 1$ (left figure) and $\alpha = 10$ (right figure). In each case, the speed c is -1.5.

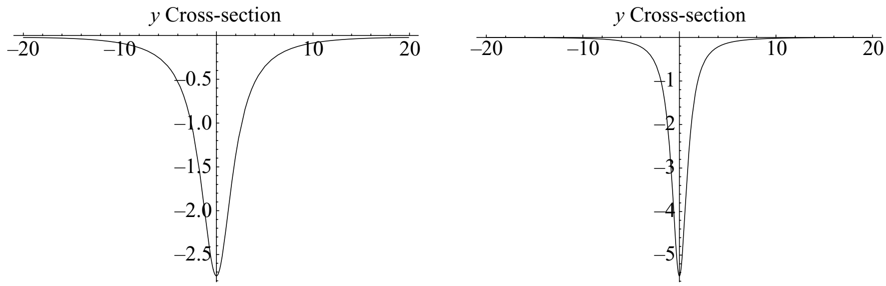

FIGURE 6. ILW–KP y cross-sections $w(0, y)$ for $\alpha = 1$ (left figure) and $\alpha = 10$ (right figure). In each case, the speed c is -1.5.

5. Conclusion

In this paper, we derived a NSP formulation for interfacial fluids with a free surface. In this formulation, the kinematic conditions in the classic two-fluid equations are replaced by a coupled system of three integral equations that depend on a free spectral parameter, and relate the free interface and surface variables. By reformulating the Bernoulli equations in terms of the interface and surface variables, a closed system is obtained that serves as an alternative formulation of the classic two-fluid equations. An advantage of the NSP equations is that the depth variable is removed.

In the first section, we presented a weak formulation of the two-fluid Euler equations. Specifically, we obtained integral equations relating the free interface and surface variables. We then analysed the connection between the NSP formulation and the Dirichlet–Neumann operators associated with the two-fluid Euler equations. We also related the weak formulation and the classic two-fluid equations. Additionally, we used the adjoint properties of the Dirichlet–Neumann operators to obtain a dual system of non-local equations. These dual equations are the formal adjoints of the non-local spectral equations, and give another integral formulation of the two-fluid problem. Finally, we demonstrated that the non-local spectral formulation captures the kinematic conditions in the two-fluid equations. To this end, we reproduced from the non-local formulation the series expansions of the Dirichlet–Neumann operators associated with the two-fluid equations.

From the NSP formulation, we obtained what we call the ILW–BL equation, a generalization of the BL equation to interfacial fluids with a free surface or rigid lid. We also derived what we refer to as the ILW–KP equation, an asymptotically distinguished (2+1) generalization of the intermediate long-wave equation, as well as higher order generalizations. We then computed lump-type solutions to the ILW–BL and ILW–KP equations, and compared their resulting speed versus amplitude curves. It is of interest that the speed–amplitude curve of the ILW–KP equation is linear, which is also the case for the KP equation.

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Appendix A. Derivation of Dirichlet–Neumann series

In Craig *et al.* (2005*a*), series expansions for the Dirichlet–Neumann operators $G(\eta)$ and $G_{kl}(\beta, \eta)$ are derived. Here we outline the derivation of the (2+1)-dimensional versions of these series from (2.33) (the derivation for $G(\eta)$ is analogous). The full details can be found in Haut (2008).

In particular, we find expansions of the form

$$G_{kl}(\eta, \beta) = \sum_{p,q=0}^{\infty} G_{kl}^{(p,q)}(\eta, \beta),$$

where the operators $G_{kl}^{(p,q)}(\eta, \beta)$ are homogenous of degree (p, q). That is, for any real numbers ϵ and δ,

$$G_{kl}^{(p,q)}(\epsilon\eta, \delta\beta) = \epsilon^p \delta^q G_{kl}^{(p,q)}(\eta, \beta). \tag{A 1}$$

To do so, first write out the matrix equation (2.33) as four equations:

$$-A_{11}(\eta)G_{11}(\eta, \beta) + A_{12}(\beta)G_{21}(\eta, \beta) = B_{11}(\eta), \tag{A 2}$$

$$-A_{21}(\eta)G_{11}(\eta, \beta) + A_{22}(\beta)G_{21}(\eta, \beta) = B_{21}(\eta), \tag{A 3}$$

$$-A_{11}(\eta)G_{12}(\eta, \beta) + A_{12}(\beta)G_{22}(\eta, \beta) = B_{12}(\beta), \tag{A 4}$$

$$-A_{21}(\eta)G_{12}(\eta, \beta) + A_{22}(\beta)G_{22}(\eta, \beta) = B_{22}(\beta). \tag{A 5}$$

Now expand the operators A_{ij} and B_{ij} about $\eta = 0$:

$$A_{11}(\eta) = \sum_{j=0}^{\infty} A_{11}^{(j)}(\eta), \quad A_{12}(\beta) = \sum_{j=0}^{\infty} A_{12}^{(j)}(\beta), \quad A_{21}(\eta) = \sum_{j=0}^{\infty} A_{21}^{(j)}(\eta),$$

$$A_{22}(\beta) = \sum_{j=0}^{\infty} A_{22}^{(j)}(\beta), \quad B_{11}(\eta) = \sum_{j=0}^{\infty} B_{11}^{(j)}(\eta), \quad B_{12}(\beta) = \sum_{j=0}^{\infty} B_{12}^{(j)}(\beta),$$

$$B_{21}(\eta) = \sum_{j=0}^{\infty} B_{21}^{(j)}(\eta), \quad B_{22}(\beta) = \sum_{j=0}^{\infty} B_{22}^{(j)}(\beta),$$

where $A_{kl}^{(j)}$ and $B_{kl}^{(j)}$ are homogenous of degree j. In particular, a straightforward calculation yields

$$A_{11}^{(j)}(\eta) = -\frac{e^{-H|D|} + (-1)^{j+1}e^{H|D|}}{j!(e^{H|D|} - e^{-H|D|})}(-1)^j|D|^j\eta^j,$$

$$A_{12}^{(j)}(\beta) = \frac{1 + (-1)^{j+1}}{2j!(e^{H|D|} - e^{-H|D|})}|D|^j\beta^j,$$

$$A_{21}^{(j)}(\eta) = -\frac{1 + (-1)^{j+1}}{2j!(e^{H|D|} - e^{-H|D|})}|D|^j\eta^j,$$

$$A_{22}^{(j)}(\beta) = \frac{e^{-H|D|} + (-1)^{j+1}e^{H|D|}}{j!(e^{H|D|} - e^{-H|D|})}|D|^j\beta^j,$$

and

$$B_{11}^{(j)}(\eta) = -\frac{e^{H|D|} + (-1)^j e^{-H|D|}}{j\,!(e^{H|D|} - e^{-H|D|})}|D|^{j-1}(D_1\eta^j D_1 + D_2\eta^j D_2),$$
$$B_{12}^{(j)}(\beta) = \frac{1 + (-1)^j}{j\,!2(e^{H|D|} - e^{-H|D|})}|D|^{j-1}(D_1\beta^j D_1 + D_2\beta^j D_2),$$
$$B_{21}^{(j)}(\beta) = -\frac{1 + (-1)^j}{j\,!2(e^{H|D|} - e^{-H|D|})}|D|^{j-1}(D_1\eta^j D_1 + D_2\eta^j D_2),$$
$$B_{22}^{(j)}(\eta) = \frac{e^{H|D|} + (-1)^j e^{-H|D|}}{j\,!(e^{H|D|} - e^{-H|D|})}|D|^{j-1}(D_1\beta^j D_1 + D_2\beta^j D_2).$$

Putting the expansions for A_{kl}, B_{kl} and G_{kl} into (A 2)–(A 5), replacing η by $\epsilon\eta$ and β by $\delta\beta$ and equating coefficients of $\epsilon^p\delta^q$ yield the desired recursion relations for $G_{kl}^{p,q}(\eta, \beta)$.

When $p + q = 0$,

$$\begin{pmatrix} G_{11}^{(0,0)}(\eta, \beta) & G_{12}^{(0,0)}(\eta, \beta) \\ G_{21}^{(0,0)}(\eta, \beta) & G_{22}^{(0,0)}(\eta, \beta) \end{pmatrix} = \begin{pmatrix} |D| \coth(|D|H) & -|D| \operatorname{csch}(|D|H) \\ -|D| \operatorname{csch}(|D|H) & |D| \coth(|D|H) \end{pmatrix}.$$

Now assuming that $p + q = m$, one can express $G_{kl}^{(p,q)}$ in terms of $G_{kl}^{(r,s)}$, when $r + s < m$. For brevity, we only consider when $p + q = m$ and $p, q \neq 0$, although the other cases are similar. From (A 2)–(A 5) we equate all terms of degree $\epsilon^p\delta^q$ to get

$$-\sum_{j=0}^{p} A_{11}^{(p-j)}(\eta)G_{11}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q} A_{12}^{(q-j)}(\beta)G_{21}^{(p,j)}(\eta, \beta) = 0,$$
$$-\sum_{j=0}^{p} A_{21}^{(p-j)}(\eta)G_{11}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q} A_{22}^{(q-j)}(\beta)G_{21}^{(p,j)}(\eta, \beta) = 0,$$
$$-\sum_{j=0}^{p} A_{11}^{(p-j)}(\eta)G_{12}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q} A_{12}^{(q-j)}(\beta)G_{22}^{(p,j)}(\eta, \beta) = 0,$$
$$-\sum_{j=0}^{p} A_{21}^{(p-j)}(\eta)G_{11}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q} A_{22}^{(q-j)}(\beta)G_{21}^{(p,j)}(\eta, \beta) = 0.$$

Using that $A_{11}^{(0)} = A_{22}^{(0)} = -\mathrm{I}$, where I is the identity operator, and $A_{21}^{(0)} = A_{12}^{(0)} = 0$, the previous four equations give us $G_{kl}^{(p,q)}$ in terms of $G_{kl}^{(r,s)}$, when $r + s < m$:

$$G_{11}^{(p,q)}(\eta, \beta) = \sum_{j=0}^{p-1} A_{11}^{(p-j)}(\eta)G_{11}^{(j,q)}(\eta, \beta) - \sum_{j=0}^{q-1} A_{12}^{(q-j)}(\beta)G_{21}^{(p,j)}(\eta, \beta), \quad \text{(A 6)}$$

$$G_{21}^{(p,q)}(\eta, \beta) = -\sum_{j=0}^{p-1} A_{21}^{(p-j)}(\eta)G_{11}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q-1} A_{22}^{(q-j)}(\beta)G_{21}^{(p,j)}(\eta, \beta), \quad \text{(A 7)}$$

$$G_{12}^{(p,q)}(\eta, \beta) = \sum_{j=0}^{p-1} A_{11}^{(p-j)}(\eta)G_{12}^{(j,q)}(\eta, \beta) - \sum_{j=0}^{q-1} A_{12}^{(q-j)}(\beta)G_{22}^{(p,j)}(\eta, \beta), \quad \text{(A 8)}$$

$$G_{22}^{(p,q)}(\eta, \beta) = -\sum_{j=0}^{p-1} A_{21}^{(p-j)}(\eta)G_{12}^{(j,q)}(\eta, \beta) + \sum_{j=0}^{q-1} A_{22}^{(q-j)}(\beta)G_{22}^{(p,j)}(\eta, \beta). \quad \text{(A 9)}$$

Finally, we need to take the adjoint equations (A 6)–(A 9) to relate them to Craig *et al.* (2005*a*). For example, taking the adjoint of (A 6) is

$$G_{11}^{(p,q)}(\eta,\beta)=\sum_{j=0}^{p-1}G_{11}^{(j,q)}(\eta,\beta)A_{11}^{(p-j)}(\eta)^{*}-\sum_{j=0}^{q-1}G_{12}^{(p,j)}(\eta,\beta)A_{12}^{(q-j)}(\beta)^{*},$$

where

$$A_{11}^{(j)}(\eta)^{*}=-\eta^{j}\frac{e^{-H|D|}+(-1)^{j+1}e^{H|D|}}{j\,!(e^{H|D|}-e^{-H|D|})}|D|^{j},$$
$$A_{12}^{(j)}(\beta)^{*}=\beta^{j}\frac{1+(-1)^{j+1}}{j\,!2(e^{H|D|}-e^{-H|D|})}|D|^{j}.$$

Appendix B. Spectral renormalization method

For concreteness, we discuss how to apply the SPRZ method to compute modes for (4.19).

First take the Fourier transform of (4.19) and rearrange the equation to get

$$\hat{w}=\frac{-(3/2)k_1^2}{(\frac{1}{2}-c^2)k_1^2+\frac{1}{2}\epsilon k_2^2+\frac{1}{2}\epsilon k_1^4-\frac{1}{4}\epsilon k_1^3\coth(k_1)}\widehat{w^2}. \tag{B 1}$$

If $1-c^2>0$, then the denominator in the previous equation is zero only when $k_1=k_2=0$.

In general, we cannot find a solution to (B 1) by naive iteration. Instead, we assume that $w=\lambda v$, where λ is an unknown parameter and v is an unknown function (this step is the renormalization part). Then (B 1) can be written in terms of λ and v as

$$\hat{v}=\lambda\frac{-(3/2)k_1^2}{(\frac{1}{2}-c^2)k_1^2+\frac{1}{2}\epsilon k_2^2+\frac{1}{2}\epsilon k_1^4-\frac{1}{4}\epsilon k_1^3\coth(k_1)}\widehat{v^2}. \tag{B 2}$$

Note that by multiplying (B 2) by $\bar{\hat{v}}$ (where $\bar{\hat{v}}$ denotes the conjugate of $\hat{v}$), rearranging, and integrating the result we get

$$\lambda=-\frac{\int_{\mathbf{R}^2}\left((\frac{1}{2}-c^2)k_1^2+\frac{1}{2}\epsilon k_2^2+\frac{1}{2}\epsilon k_1^4-\frac{1}{4}\epsilon k_1^3\coth(k_1)\right)\hat{v}\bar{\hat{v}}\,dk}{\int_{\mathbf{R}^2}(3/2)k_1^2\widehat{v^2}\bar{\hat{v}}\,dk}. \tag{B 3}$$

Finally, we use (B 2) and (B 3) for our SPRZ scheme:

$$\widehat{v_{n+1}}=-\lambda_{n+1}\frac{(3/2)k_1^2}{(\frac{1}{2}-c^2)k_1^2+\frac{1}{2}\epsilon k_2^2+\frac{1}{2}\epsilon k_1^4-\frac{1}{4}\epsilon k_1^3\coth(k_1)}\widehat{{v_n}^2}, \tag{B 4}$$

$$\lambda_{n+1}=-\frac{\int_{\mathbf{R}^2}\left((\frac{1}{2}-c^2)k_1^2+\frac{1}{2}\epsilon k_2^2+\frac{1}{2}\epsilon k_1^4-\frac{1}{4}\epsilon k_1^3\coth(k_1)\right)\widehat{v_n}\overline{\widehat{v_n}}\,dk}{\int_{\mathbf{R}^2}(3/2)k_1^2\widehat{{v_n}^2}\overline{\widehat{v_n}}\,dk}. \tag{B 5}$$

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